10.1 First Order Ordinary Differential Equations

1164. Linear Equations

$$\frac{dy}{dx} + p(x)y = q(x).$$

The general solution is

$$y = \frac{\int u(x)q(x)dx + C}{u(x)},$$

where

$$u(x) = \exp\left(\int p(x)dx\right).$$

1165. Separable Equations

$$\frac{dy}{dx} = f(x,y) = g(x)h(y)$$

The general solution is given by

$$\int \frac{dy}{h(y)} = \int g(x)dx + C,$$

or

$$H(y) = G(x) + C.$$

1166. Homogeneous Equations

The differential equation $\frac{dy}{dx}=f(x,y)$ is homogeneous, if the function $f(x,y)$ is homogeneous, that is
$f(tx,ty)=f(x,y)$.

The substitution $z=\frac{y}{x}$ (then $y=zx$) leads to the separable equation
$$x\frac{dz}{dx}+z=f(1,z).$$

1167. Bernoulli Equation
$$\frac{dy}{dx}+p(x)y=q(x)y^n.$$

The substitution $z=y^{1-n}$ leads to the linear equation
$$\frac{dz}{dx}+(1-n)p(x)z=(1-n)q(x).$$

1168. Riccati Equation
$$\frac{dy}{dx}=p(x)+q(x)y+r(x)y^2$$

If a particular solution y_1 is known, then the general solution can be obtained with the help of substitution
$z=\frac{1}{y-y_1}$, which leads to the first order linear equation
$$\frac{dz}{dx}=-[q(x)+2y_1r(x)]z-r(x).$$

1169. Exact and Nonexact Equations

The equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

and nonexact otherwise.

The general solution is

$$\int M(x,y)dx + \int N(x,y)dy = C.$$

1170. Radioactive Decay

$$\frac{dy}{dt} = -ky,$$

where $y(t)$ is the amount of radioactive element at time t, k is the rate of decay.

The solution is

$y(t) = y_0 e^{-kt}$, where $y_0 = y(0)$ is the initial amount.

1171. Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - S),$$

where $T(t)$ is the temperature of an object at time t, S is the temperature of the surrounding environment, k is a positive constant.

The solution is

$$T(t) = S + (T_0 - S)e^{-kt},$$

where $T_0 = T(0)$ is the initial temperature of the object at time $t = 0$.

1172. Population Dynamics (Logistic Model)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where $P(t)$ is population at time t, k is a positive constant, M is a limiting size for the population.

The solution of the differential equation is

$P(t) = \dfrac{MP_0}{P_0 + (M - P_0)e^{-kt}}$, where $P_0 = P(0)$ is the initial population at time $t = 0$.